QCD Instantons in Vacuum and Matter

I. Zahed

Department of Physics
SUNY, Stony Brook, New York 11794

Abstract

The singlet coupling to the topological charge density in the instanton vacuum, causes the instantons and antiinstantons to be screened over distances of the order of 1/2 fm. Dilute instanton systems behave as a free gas, while dense instanton systems behave as a plasma. The free gas behaviour is favored by a density of 1 fm^{-4}. Owing to the Higgs mechanism, the η' mass is heavy (1100 MeV). The vacuum topological susceptibility is small (0.07 fm^{-4}) and consistent with the QCD Ward identity. In the chiral limit, the singlet screening vanishes, leading to a four dimensional plasma state with a temperature given by the nonet decay constant f (90 MeV). The phase is Debye screened. In the presence of matter, the screening is quenched, and a Kosterlitz-Thouless transition may occur signaling the restoration of the $U_A(1)$ symmetry. We suggest to use the spatial asymptotics of the static topological charge correlator together with the topological susceptibility to probe the interplay between the $U_A(1)$ restoration and the chiral restoration in finite temperature QCD. The relevance of these results to the bulk thermodynamics in the instanton vacuum is discussed.

1. Introduction

The instanton approach to the QCD ground state has received continuous attention in the past few years [1, 2, 3, 4]. Overall, it provides a fair description of the bulk vacuum properties and hadronic correlations. Recent (quenched) lattice simulations [5] have suggested that the bulk structure and hadronic correlations are mildly affected by cooling, an indication that semiclassical physics may be operative in the long wavelength limit.

Extensive calculations using instantons have been carried out in the quenched approximation (equivalently large N_c limit), both analytically [6, 7] and numerically [4]. While the numerical calculations were found to be consistent with the cooled lattice simulations, these results suggest that the instantons in the vacuum state are in a dilute gas approximation. To the annoying exception of the η' mass, which comes out to be too heavy in the numerical instanton calculations, the overall part of the hadronic spectrum is reproduced by fitting the pion decay constant and the fermion condensate. The model does not confine, yet it breaks spontaneously chiral symmetry with the generation of a momentum dependent constituent mass.

To understand some of the features of the model, it suffices to recall few thinghs. The pseudoscalar nonet follows from symmetry provided that the pion decay constant and the fermion condensate are reasonable, given suitable current quark masses. In the dilute gas approximation, the vector and isobar correlators are not affected by instantons at small distances. The formers because of the self-duality character of the instanton configuration, and the latters because of the flavor character of the induced instanton interaction. Interesting effects in the nucleon channel have been reported suggesting diquark correlations at intermediate distances [4]. Since the model lacks confinement, the large distance behaviour of these correlators is dominated by constituent quark thresholds.

A major unknown in instanton calculations resides in the character of the instanton-antiinstanton interaction. Earlier calculations based on variational estimates with a sum ansatz in the quenched approximation, have led to a reasonable description of the bulk structure [8]. However, these estimates were found to be quantitativaly sensitive to the short distance character of the interaction. The latter was ansatz dependent. More elab-

2

orated numerical analyses, based on stream-line configurations as the 'best' instanton-antiinstanton prototypes, have shed question marks on the use of a semiclassical approximation altogether [9]. However, these configurations hold only in the quenched approximation, where the instanton-antiinstanton interaction is dipole-like at large distances. They may not be important in the unquenched approximation, where screening by charges in the fundamental representation is in effect.

Semiclassical physics in QCD has been challenged by Greensite [10] using a combination of arguments based on large N_c, factorization and the master field equation. Greensite's objections to instantons (and generically to semiclassical models based on statistical ensembles) may be evaded by noting that the large N_c limit does not commute with the thermodynamical limit.

In this paper we would like to come back to the annoying problem in the η' channel. Since instantons have been professed to cure the $U_A(1)$ problem [11], this discrepancy cannot be ignored. Also, in the unquenched approximation, the instanton-antinstanton interaction is not dipole-like at large distances, but screened over distance scales of the order of 1/2 fm. This screening is caused by the mixing between the singlet and the topological density in the vacuum, and is at the origin of the large η' mass, thus resolving the $U_A(1)$ problem. In fact this construction was briefly presented in [7] and more recently discussed in the context of the strong CP problem [12]. We will also show that the resulting vacuum topological susceptibility is consistent with the QCD Ward identity. In the unquenched approximation, dilute systems of instantons behave as a free gas. Throughout, the short distance part of the instanton-antiinstanton interaction will be assumed. In the presence of matter the screening decreases. Our approach suggests a dual description for the bulk pressure : either as a free meson gas with a heavy η', or as a free meson gas with a light singlet plus a screened Coulomb gas of instantons and antiinstantons. At temperatures for which the screening length becomes comparable to the interparticle distance, a Kosterlitz-Thouless transition [13] may occur. This transition may be related to the $U_A(1)$ restoration in QCD. We suggest to use the large spatial asymptotics of the topological charge correlator to account for the restoration of $U_A(1)$ in the QCD ground state. We explicitly show that the bulk thermodynamics at low temperature is meson dominated, and a rapid crossover in the topological susceptibity

is expected with the vanishing of the fermion condensate. Throughout, the octet-singlet mixing will be ignored and the vacuum angle will be set to zero.

2. Model

Few years ago, Nowak, Verbaarschot and I [7] suggested that in the long wavelength approximation the salient features of the instanton model to the QCD ground state follows from symmetries and anomalies. If we were to denote by $n^{\pm}$ the local densities of instantons and antiinstantons (thought about here as quasiparticles), then the partition function in the presence of a pseudoscalar source term S^a, reads [7]

$$Z[S] = \int dn^{\pm} \int dK \ e^{-W[K]-W_I[K,n]+\int d^4x \text{Tr}(SK)} \tag{1}$$

where $W[K]$ is the pseudoscalar effective action

$$W[K] = \int d^4x \left(\frac{1}{2}(\partial_\mu K^a)^2 - \frac{1}{2}\frac{<\overline{\psi}\psi>}{f^2} \text{Tr}(mK^2)\right) \tag{2}$$

and $W_I[K,n]$ is the singlet coupling to the local topological charge $(n^+ - n^-) \sim E^a \cdot B^a$,

$$W_I[K,n] = -i \int d^4x \frac{\sqrt{2N_F}}{f}(n^+ - n^-) \ K^0(x) \tag{3}$$

We have defined the nonet by $K = \sum_{a=0}^{8} K^a \lambda^a$, with the normalization $\text{Tr}(\lambda^a \lambda^b) = \delta^{ab}$. The mass matrix $m = \text{diag}(m, m, m_s) \sim \text{diag}(5, 5, 120)$ MeV. The imaginary part follows from the fact that (3) is T-odd in Minkowski space. The singlet coupling to the topological charge is $\sqrt{2N_F}/f$, where $f \sim 90$ MeV is the bare nonet decay constant, and N_F the number of flavours. It vanishes in the large N_c limit as it should. The fermion condenstate $<\overline{\psi}\psi> \sim -(255 \text{ MeV})^3$, for an instanton density $<N>/V_4 \sim 1$ fm^{-4}. We arrived at (1) by integrating over a random system of instantons and antiinstantons and then bosonizing the multi-t' Hooft vertices. We note that eventhough W_I is subleading in $1/N_c$, since the instanton-antiinstanton density $<N>/V_4$ is of order N_c, a statistical averaging of this part leads effects of order N_c^0 (see below), thus comparable to the free meson action W.

4

If we were to drop this term, the present analysis is totally in agreement with the detailed analysis of Diakonov and Petrov [1], in the two flavour case. This term reflects on the fermionic induced interactions and will be important for the discussion below. We have explicitly disregarded the scalar fields for convenience. The latters are relevant for the issue of the scale anomaly and the bulk compressibility. We refer to [7] for a discussion of these issues. We stress that the fields K are auxillary (integrated over), while the physical meson sources are represented by S. The auxillary fields arise from the bosonization of the multiflavour quark interactions triggered by a random instanton gas. The *a posteriori* legitimacy of the random gas approximation will be discussed below.

To leading order in N_c and in the long wavelength limit, the fermionic correlation functions follow from (3). Specifically

$$
\begin{aligned}
\langle \overline{\psi} i\gamma_5 \lambda^a \psi(x) \overline{\psi} i\gamma_5 \lambda^a \psi(0) \rangle \sim \quad & -N_c \mathrm{Tr}\left(\lambda^a S_F^+(x) \lambda^a S_F(-x) \right) \\
& + \frac{<\overline{\psi}\psi>^2}{f^2} \left(\frac{\delta^2 \mathrm{ln}Z[S]}{\delta S^a(x)\delta S^a(0)} \right)_{S=0}
\end{aligned}
\tag{4}
$$

where $S_F(x)$ is the constituent quark propagator, and the trace is over the spin variables. It suffices to know, that asymptotically $S_F(x) \sim M^2 e^{-M|x|}/|x|$, where $M \sim m + M(0) \sim m + 345$ MeV [1, 7]. Similar expressions can be derived in other channels. The presence of the two-constituent quark cut in (4) reflects on the lack of confinement in the model. It is interesting to compare expression (4) with a similar expression derived in two dimensional QCD for large N_c [14]. In the latter, the two-fermion cut is infrared sensitive and cancels precisely against the infrared sensitive one-gluon exchange graph. The result is a sum of mesonic poles only. This cancellation makes explicit use of Ward identities in Feynman graphs. It is quantum, and thus absent in the semiclassical approach followed here.

For finite N_c the fermions feed back on the instantons and vice-versa. A straightforward way to incorporate this, is to use (1) for finite N_c. This is an approximation, as subleading parts in the multi-'t Hooft interactions were disregarded in (1). We expect these parts to renormalize the fermion condensate and the nonet decay constant. With this in mind and in the long wavelength limit, we can approximate the instantons and anti-instantons by point like structures (the effects of their sizes will be discussed below) and define the topological charge to be ($N = N_+ + N_-$)

$$(n^+ - n^-)(x) = \sum_{i=1}^{N} Q_i \delta_4(x - x_i) \tag{5}$$

with $Q_\pm = \pm 1$. Thus, the interaction term between instantons in the long wavelength limit is mostly triggered by the singlet K^0, through their topological charge

$$W_I = -i\frac{\sqrt{2N_F}}{f} \sum_{i=1}^{N} Q_i \; K^0(x_i) \tag{6}$$

We note that the combination $(i\sqrt{f}K^0)$ plays the role of a four dimensional Coulomb potential (the i for Euclidean), with Coulomb charges $q_i = Q_i\sqrt{2N_F/f}$. This point will be further discussed below. If we denote by $z_\pm$ the fugacity of the quasiparticles, then substituting (5) in (1) gives ($N = N_+ + N_-$)

$$Z[S] = \sum_{N_\pm} \frac{z_+^{N_+} z_-^{N_-}}{N_+! N_-!} \prod_{i=1}^{N} \int d^4x_i \int d[K] \; e^{-W[K]+i\frac{\sqrt{2N_F}}{f}\sum_{i=1}^{N} Q_i \; K^0(x_i)+\int d^4x \, \text{Tr}(SK)} \tag{7}$$

The fugacities follows from $< N_\pm > = \partial \ln z_\pm Z[0]/\partial z_\pm$. For a noninteracting system $\ln Z[0] \sim V_4(z_+ + z_-)$. Thus $z_\pm \sim (< N > /2V_4)$.

In the vacuum ($S = 0$) we can rewrite (7) in two equivalent ways. By summing over the instanton-antiinstanton degrees of freedom in the long-wavelength limit, leaving out the singlet field. Thus

$$Z[0] = \int d[K] \; e^{-W[K]+\int d^4x \, 2z \, \cos(\sqrt{2N_F}K^0/f)} = \int d[K] \; e^{-S[K]} \tag{8}$$

which shows that $Z[0]$ is a sum of zero point contributions from a free but light octet of pseudoscalars, and a self-interacting but heavy pseudoscalar singlet. This version of $Z[0]$ will be relevant for the discussion of matter effects below. Equivalently, we can choose to integrate over the singlet K^0 in (7) leaving out the instanton-antiinstanton degrees of freedom . Ignoring singlet-octet mixing, the result is

$$Z[0] = \left(\int d'[K]e^{-W[K]}\right) \sum_{N_\pm} \frac{z^{N_+ + N_-}}{N_+! N_-!} \prod_{i=1}^{N} \int d^4x_i \; e^{-N_F \frac{m_0^2}{f^2} \sum_{i,j} Q_i Q_j \left(\frac{1}{2\pi^2} \frac{\mathbf{K}_1(m_0|x_i - x_j|)}{m_0|x_i - x_j|}\right)} \tag{9}$$

6

where $m_0^2 = 2/3m_K^2 + 1/3m_\pi^2$ is the singlet K^0 mass, and $\mathbf{K}_1$ a Bessel function. The singlet K^0 field is excluded from the mesonic measure, thus the prime. (9) is the product of the vacuum partition function of free massive octets and a four dimensional Coulomb gas with Yukawa interactions. A similar result was also obtained by Kikuchi and Wudka [12]. The mixing (7) causes the instanton-antinstantons in the vacuum to be screened over distances of the order of $1/m_0 \sim 1/2$ fm. Note that the partition function for the Coulomb gas is ill-defined as $x_i \to x_j$. Throughout, we will assume that the instantons and antiinstantons have a *core* (smeared). The effects of the smearing will cause the fugacities to renormalize by self-interaction terms. In the long wavelength limit, the physics is insensitive to the detailed choice of the core.

The mixing (6) causes the η' excitation to be heavier than the rest of the octet. This is just the Higgs mechanism. Indeed, the pseudoscalar correlator correlator follows from

$$Z[\eta'] = \int d[K] e^{-W[K] + \int d^4x \, 2z \, \cos(\sqrt{2N_F}K^0/f) + \int d^4x \eta' \, K^0/\sqrt{2N_F}} \tag{10}$$

using (4). In the Gaussian approximation (order N_c^0 in the action and N_c in the correlator) the result is

$$\langle \overline{\psi} i\gamma_5 \psi(x) \overline{\psi} i\gamma_5 \psi(0) \rangle \sim -N_c \text{Tr} \left(S_F^+(x) \, S_F(-x) \right) + \frac{<\overline{\psi}\psi>^2}{2N_F f^2} \left(\frac{m_{\eta'}^2}{2\pi^2} \left(\frac{\mathbf{K}_1(m_{\eta'}|x|)}{m_{\eta'}|x|} \right) \right) \tag{11}$$

where the square of the η' mass is given by

$$m_{\eta'}^2 = m_0^2 + 2z \frac{2N_F}{f^2} \tag{12}$$

which is the result quoted in [7]. For $N_F = 3$, $f \sim 90$ MeV and $z \sim 1/2$ fm^{-4} we have $m_{\eta'} \sim 1100$ MeV. The present analysis is consistent with the analysis suggested by Witten [16] and Veneziano [17], except for one thingh : $z \sim N_c$, so that the contribution of the Coulomb gas is $z/f^2 \sim N_c^0$. The η-η' splitting does not vanish in the large N_c limit in the instanton vacuum. Should we be concerned ? Yes, if we were to believe that the transition from $N_c = 3$ to $N_c = \infty$ is smooth in QCD (no phase change). Phase changes may occur in instanton systems with large values of N_c [8]. For a dilute system $N/V_4 \sim 2z \sim 1$ fm^{-4},

the interparticle distance is about 1 fm, which is twice the screening length $1/m_0 \sim 1/2$ fm. In this regime, the instantons and antiinstantons behave as a free noninteracting gas. This result shows that the approximations used in [3, 4, 6, 7] in which the instanton and antiinstanton distributions were taken to be totally random is justified in the unquenched approximation (finite N_c). Finally, the induced term (7) causes the η' to self-interact. The η' quartic interaction term has a strength of order $2z(\sqrt{2N_F}/f)^4 \sim 10^2$, which is strong. We note that this interaction vanishes in the large N_c limit. Finally, we note that the η' pole in (11) lies above the two-constituent quark cut, which is about 700 MeV, leaving the η' with a broad width. This problem is generic to models that do not confine.

3. T = 0 Susceptibilities

The topological susceptibility of the vacuum can be calculated using

$$\chi(x - y) = \left\langle \sum_i Q_i \delta_4(x - x_i) \sum_j Q_j \delta_4(y - x_j) \right\rangle \tag{13}$$

since $E \cdot B \sim (n^+ - n^-)$. The expectation value (13) follows from the following generating functional

$$Z[\theta] = \sum_{N_\pm} \frac{z^{N_+ + N_-}}{N_+! N_-!} \prod_{i=1}^N \int d^4 x_i \; e^{Q_i \theta(x_i)} \; e^{-N_F \frac{m_0^2}{f^2} \sum_{i,j} Q_i Q_j \left(\frac{1}{2\pi^2} \frac{\mathbf{K}_1(m_0 |x_i - x_j|)}{m_0 |x_i - x_j|} \right)} \tag{14}$$

by taking $\partial^2 \ln Z / \partial \theta(x) \partial \theta(y)$ and setting θ to zero. A straighforward calculation gives

$$\chi(x - y) = \frac{f^2}{2N_F}(m_{\eta'}^2 - m_0^2) \left(\delta_4(x - y) - (m_{\eta'}^2 - m_0^2) \frac{m_{\eta'}^2}{2\pi^2} \left(\frac{\mathbf{K}_1(m_{\eta'}|x - y|)}{m_{\eta'}|x - y|} \right) \right) \tag{15}$$

At large separations, the susceptibility falls off exponentially at a rate given by the physical η' mass. This fall off agrees with numerical simulations [15]. We note that (15) is analogous to the result in the two dimensional Schwinger model, where the susceptibility is given by the correlation function of the electric field [18]

$$\chi_S(x - y) = \frac{1}{4\pi^2} < E(x)E(y) > = \frac{1}{4\pi^2} \left(\delta_2(x - y) - \frac{m_S^2}{2\pi} \mathbf{K}_0(m_S|x - y|) \right) \tag{16}$$

8

with $m_S = e/\sqrt{\pi}$ the photon screening mass and $\mathbf{K}_0$ a Bessel function. The topological susceptibility obeys a zero momentum sum rule. Indeed, Fourier transforming (15) gives at zero momentum

$$\chi(q = 0) = -\frac{2m + m_s}{3} \frac{<\overline{\psi}\psi>}{2N_F} \left(1 + \frac{2m + m_s}{3} \frac{<\overline{\psi}\psi>}{f^2 m_{\eta'}^2} \right) \tag{17}$$

in agreement with the QCD Ward identity [19]

$$\int d^4x \, \langle G(x)G(0) \rangle = -m \frac{<\overline{\psi}\psi>}{2N_F} + \frac{m^2}{4N_F^2} \int d^4x \, \langle \overline{\psi}\gamma_5\psi(x)\overline{\psi}\gamma_5\psi(0) \rangle \tag{18}$$

where we have used $G = (g^2/16\pi^2)\vec{E}^a \cdot \vec{B}^a$. For the parameters quoted above, $\chi \sim .07$ fm^{-4}, which is small. Note that in the chiral limit, the topological susceptibility (17) vanishes like in the Schwinger model where $\chi_S(q = 0) = 0$.

By analogy with the Schwinger model, the following expectation value

$$\langle e^{-i \int_{D_4} d^4x G(x)} \rangle = < \prod_i^N e^{-iQ_i \int_{D_4} d^4x \, \delta_4(x - x_i)} > \tag{19}$$

can be estimated. Here D_4 is a four dimensional volume with $D_3 = \partial D_4$ as a border. This expectation value reflects on the amount of topological screening in the vacuum. Smilga [20] has suggested that (19) obeys a four-volume law in Yang-Mills theories, and a three-surface law in QCD. In the Schwinger model the screening is total, and the analogue of (19) is the Wilson loop. The latter obeys a perimeter law ($\chi_s(q = 0) = 0$). In our case, a semiclassical estimate of (19) gives

$$< \prod_i^N e^{-iQ_i \int_{D_4} d^4x \, \delta_4(x - x_i)} > \sim e^{-S[K_{cl}^0]} \tag{20}$$

where K_{cl}^0 is the classical solution to

$$(-\Box + m_0^2)K_{cl}^0(x) + 2z\frac{\sqrt{2N_F}}{f}\sin\left(\frac{\sqrt{2N_F}}{f}K_{cl}^0(x)\right) = \frac{f}{\sqrt{2N_F}}(-\Box + m_0^2)\int_{D_4} d^4y \, \delta_4(x - y) \tag{21}$$

for a strongly coupled source. While we do not know of a general solution to (21) in four dimensions, in the linear approximation (21) can be readily solved. In this case, the solution is equivalent to taking the second cumulant in (19). The latter is just the topological susceptibility,

$$< \prod_i^N e^{-iQ_i \int_{D_4} d^4x \, \delta_4(x-x_i)} > \sim e^{-\frac{D_4}{2}\chi(q=0)} \tag{22}$$

Since (17) does not vanish, the second cumulant indicates that (19) obeys a four-volume law. The fall off rate is $\chi/2 \sim 10^{-2}$ fm^{-4}. This result is not exact and maybe affected by higher cumulants (non-Gaussian fluctuations). We note that the cumulant expansion is consistent with the dilute gas approximation. In the chiral limit, the topological susceptibility vanishes, and the screening is total. Indeed, for $m = m_s = 0$ (22) falls off with the border $D_3 = \partial D_4$

$$< \prod_i^N e^{-iQ_i \int_{D_4} d^4x \, \delta_4(x-x_i)} > \sim e^{-\frac{1}{2}(\frac{zm_*^2}{\pi^2}) \int_{D_3 \times D_3} d\vec{\Sigma}_x \cdot d\vec{\Sigma}_y \left(\frac{\mathbf{K}_1(m_*|x-y|)}{m_*|x-y|} \right)} = e^{-(\frac{2z}{m_*})D_3} \tag{23}$$

where $m_*^2 = 4N_F z/f^2$ with $d\vec{\Sigma}$ the three-normal to the four-volume D_4. The fall-off rate is given by

$$\frac{2z}{m_*} = \frac{f^2}{2N_F} \left(m_{\eta'}^2 - \frac{2}{3}m_K^2 - \frac{1}{3}m_\pi^2 \right)^{1/2} \tag{24}$$

which is about 1 fm^{-3}. (23) follows from the second cumulant and large three-surfaces $D_3 m_*^3 >> 1$. This result is analogous to the result in the Schwinger model. In the chiral limit, the instantons and antinstantons are in a strongly coupled plasma-like phase driven by a Debye-Huckel equation, with an effective temperature $T_* \sim f$ (see below).

The fluctuations in the number density of instantons and antiinstantons can be obtained using similar arguments, if we were to note that

$$n^+ + n^- = \sum_i^N \delta_4(x - x_i) \tag{25}$$

Since we have ignored the scalars we will only provide a simple estimate in this case. A straighforward calculation gives

$$\sigma(x-y) \sim \langle \sum_{i=1}^{N} \delta_4(x-x_i) \sum_{i=j}^{N} \delta_4(y-x_j) \rangle_C \sim 2z\delta(x-y) \tag{26}$$

where only the connected part has been retained. This result shows that the variance in the number of particles is Poissonian, and that the compressibility is of order 1 fm^{-2}. The inclusion of the scalars smears the correlations (26) over the scalar Compton wavelength, causing the compressibility to increase. These effects along with the QCD scale Ward identities [21] will be discussed elsewhere [22].

The present interplay between the instantons and the η' with the subsequent screening, does not affect the pseudoscalar octet in the long wavelength approximation. Indeed, from (1) it follows that the auxillary meson fields decouple from the topological charge to order N_c^0, to the exception of the singlet K^0 (ignoring singlet-octet mixing). A self-consistent calculation would require that instead of using a random gas approximation, we should use a screened Coulomb gas approximation, e.g. (14). However, the diluteness of the system implies that the effects are small, since the interparticle distance is about twice the screening length. Thus we would predict that the fermion condensate and the nonet decay constant will not be affected considerably by the screening mechanism. In a way, this is good because it implies that most of the calculations performed in the random gas approximation reflect correctly on unquenched QCD [4, 6, 7].

4. $T \neq 0$ Susceptibilities

How does the temperature affects the present arguments ? First, consider the correlation in the topological charge (13). At low temperature, the system is meson dominated. The temperature enters both through the meson parameters [1] as well as the meson distributions. Thus $(\omega_n = 2\pi nT)$

$$\chi(x,y;T) = \quad +\frac{f^2(T)}{2N_F}(m_{\eta'}^2(T) - m_0^2(T))\delta_4(x-y) \tag{27}$$
$$-\frac{Tf^2(T)}{4N_F}\frac{(m_{\eta'}^2(T) - m_0^2(T))^2}{|\vec{x}-\vec{y}|} \sum_{n=-\infty}^{+\infty} e^{i\omega_n(x^0-y^0) - \sqrt{\omega_n^2 + m_{\eta'}^2(T)}|\vec{x}-\vec{y}|}$$

---

[1] The fermion constituent mass becomes temperature dependent [23, 24].

The static part $\chi(\omega = 0, \vec{x}; T)$ of (27) is dominated by the η' excitation

$$
\begin{aligned}
\chi(\omega = 0, \vec{x}; T) = \quad & T \int_0^{\frac{1}{T}} dx^0 \chi(x, 0; T) \\
= \quad & \frac{T f^2(T)}{2N_F} (m_{\eta'}^2(T) - m_0^2(T)) \left(\delta_3(x) - \frac{(m_{\eta'}^2(T) - m_0^2(T))}{2|\vec{x}|} e^{-m_{\eta'}(T)|\vec{x}|} \right)
\end{aligned}
\tag{28}
$$

In this form, the QCD Ward identity is fulfilled even at finite temperature,

$$
\int d^3 x \chi(\omega = 0, \vec{x}, T) = -\frac{2m + m_s}{3} \frac{<\overline{\psi}\psi>(T)}{2N_F}
\tag{29}
$$

and vanishes at the chiral transition point. This, however, does not necessarily mean that the $U_A(1)$ symmetry is restored. The restoration of the latter can be asserted only through the large distance behaviour of (28). In the $U_A(1)$ broken phase, the correlator falls off exponentially, while in the symmetric phase it vanishes identically. This point is worth probing both on the lattice and in the numerical instanton calculations [4]. For completeness, we note that the temporal asymptotics of (28) is free field dominated for temperatures $T \sim m_{\eta'}/2\pi \sim 150$ MeV ($0 < x^0 < 1/T$). The compressibility $\sigma(T) \sim 2z(T)$, and is expected to drop by $1/2$ at the chiral transition point, following the drop in the gluon condensate [25]. Debye screening at high temperature [26] causes the electric condensate to vanish. In instanton models that would also mean the vanishing of the magnetic condensate because of self duality.

In the Schwinger model, the temperature effects on the topological correlator can be calculated exactly. Indeed, the correlation in the topological charge at finite temperature is given by

$$
\chi_S(x, T) = \frac{1}{4\pi^2} \left(\delta_2(x) - \frac{T m_S^2}{2\pi} \sum_{n=-\infty}^{+\infty} \frac{1}{\sqrt{\omega_n^2 + m_S^2}} e^{i\omega_n x^0} e^{-\sqrt{\omega_n^2 + m_S^2}|x^1|} \right)
\tag{30}
$$

which is the analogue of (27). The analogue of the static part of the susceptibility (28) is given by

$$
\chi_S(\omega = 0, x^1; T) = \int_0^{\frac{1}{T}} \chi_S(x, T) = \frac{T}{4\pi^2} \left(\delta(x^1) - \frac{m_S}{2} e^{-m_S|x^1|} \right)
\tag{31}
$$

whatever T. We observe that eventough (31) integrates to zero, the correlator still falls off exponentially. In the Schwinger model m_S follows directly from the $U_A(1)$ anomaly (bubble graph) and thus is T-independent for all temperatures. This is not the case for $m_{\eta'}$ as we have discussed above. For completeness, we also note that along the temporal direction (30) is dominated by the free field behaviour for temperatures of the order of $T \sim m_S/2\pi$ $(0 < x^0 < 1/T)$,

$$\chi_S(x^0, T) \sim \frac{1}{4\pi^2}\left(V_1\delta(x^0) - \frac{Tm_S}{2\pi} + \frac{m_S^2}{2\pi^2}\ln(2\pi T|x^0|)\right) \tag{32}$$

where V_1 is the space length.

5. Pressure

At low temperature, the pressure is just given by the one loop effects following from (1) and the rest of the mesonic action. Thus

$$\mathbf{P} + \mathbf{B} = \mathbf{P}_\pi + \mathbf{P}_K + \mathbf{P}_\eta + \mathbf{P}_{\eta'} + ... \tag{33}$$

where $\mathbf{B}$ is the vacuum pressure, and $\mathbf{P}_a$ the respective mesonic pressures. The dots stand for the higher hadron resonances, in agreement with the Gibbs average. The mesons in (33) carry temperature dependent masses, since the fermion condensate is temperature dependent. At low temperature, the effects are small, and the pressures $\mathbf{P}_a$ in (33) are the usual black-body contributions. Below the deconfining transition, the vacuum pressure is T-independent in a confining theory [27], to leading order in N_c. This is not the case here and $\mathbf{B}$ receives an unwanted contribution from the free constituent quark loop of order N_c.

In writing (33) we have ignored the η' self-interactions, $i.e.$ we have kept the order N_c^0 terms in the free energy. For finite N_c these effects, may not be small. Indeed, strong self-interactions may give rise to classical solutions to $(\phi = i\sqrt{f}K^0)$

$$\left(-\Box + m_0^2\right)\phi = -z\left(\frac{2N_F}{f}\right)^{1/2}\left(e^{-\frac{\sqrt{2N_F/f}}{f}\phi} - e^{+\frac{\sqrt{2N_F/f}}{f}\phi}\right) \tag{34}$$

which shows that ϕ (the precursor of the η') plays the role of a coarse grained Coulomb potential in four dimensions. In the chiral limit ($m_0 = 0$) (34) is the Debye-Huckel equation for a four dimensional Coulomb plasma with charges $q = \sqrt{2N_F/f} \sim 1/\sqrt{N_c}$, charge density $n \sim zq/4\pi \sim \sqrt{N_c}$, and an effective temperature $T_* \sim f \sim \sqrt{N_c}$ [2]. For large N_c, the Coulomb plasma is classical, and (33) follows. The $\eta - \eta'$ mass difference is just the Debye screening length. At high temperature, the effects of the dropping masses may be important. The derivation, however, becomes less reliable as other effects (quantum fluctuations, subleading $1/N_c$ terms, ...) may be important. Note that at high temperature the four dimensional Debye-Huckel equation (34) (taken litterally) dimensionally reduces to an equation in three dimensions. The effective temperature for the three dimensional Coulomb gas is $f \sim 0$, that is low. The system has a tendency to cluster. This tendency is perhaps what has been observed by Ilgenfritz and Shuryak [28] in interacting instanton systems with fermion determinants. In a way the high temperature Coulomb gas problem discussed here behaves as a low temperature Kosterlitz-Thouless system [13]. Coulomb systems in two dimensions display a dipole phase at low temperature, and a plasma phase at high temperature. If such a transition were to occur, it is plausible that the resulting phase is $U_A(1)$ symmetric. Since $\chi(0) \sim m < \overline{\psi}\psi >$, it is possible that this phase change may coincide with the chiral transition. This point can be elucidated by lattice simulations that will calculate both $\chi(\omega = 0, \vec{q} = 0, T)$ and $\chi(\omega = 0, |\vec{x}|, T)$, as suggested above.

We note that the effects of the instantons in (33) is implicit, and manifest through the larger η' mass even at finite temperature (ignoring η' interactions). An equivalent way of describing the same physics, is to rewrite (33) as follows

$$\mathbf{P} + \mathbf{B} = \mathbf{P}_\pi + \mathbf{P}_K + \mathbf{P}_\eta + (\mathbf{P}_{K^0} + P_I) + ... \tag{35}$$

where P_{K^0} is the contribution to the pressure coming from a thermalized system of singlet mesons with mass m_0, and P_I the instanton contribution following from the finite temperature partition function

---

[2] A four dimensional plasma, follows from a five dimensional field theory with $\phi \sim M^{3/2}$ and $q \sim M^{-1/2}$ where M carries canonical mass dimension.

$$Z_I[T] = \sum_{N_\pm} \frac{z^{N_+ + N_-}}{N_+! N_-!} \prod_{i=1}^{N} \int_{R^3 \times 1/T} d^4 x_i \ e^{-N_F \frac{T}{f^2(T)} \sum_{i,j,n} Q_i Q_j \int \frac{d^3 q}{(2\pi)^3} \frac{e^{iq \cdot (x_i - x_j)}}{q_n^2 + m_0^2(T)}} \tag{36}$$

where $q_n = (2\pi n T, \vec{q})$. $P_{\eta'} = P_{K^0} + P_I$. The Coulomb gas description referred to above is now manifest. This decomposition, allows us to see how matter affects instantons. It also shows the artificial character of the decomposition in (35). At low temperature, the condensate (thus m_0) and the meson decay constant f do not change appreciably. The instanton screening length remains about the same. At high temperature m_0 is expected to drop, weakening the screening. Our calculations break down when the screening becomes comparable to the interparticle distance, *i.e.* $m_0^4 \sim < N > T/V_3$. Subleading terms in $1/N_c$ and the omitted scalars as well as higher lying hadrons are important at high temperature.

6. Conclusions

We have given some qualitative arguments indicating the interplay between the η' fluctuations and the instantons in the vacuum state and in matter. At zero temperature, the instantons are screened over distances of the order of 1/2 fm, and the η' acquires a mass of the order of 1100 MeV, leading to a natural resolution of the $U_A(1)$ problem, as originally suggested by 't Hooft [11]. These results emphasize our earlier work [7] and are in general agreement with the arguments given by Kikuchi and Wudka [12]. The screening among the instantons and antiinstantons in the vacuum yields a topological susceptibility that is consistent with the QCD Ward identity. For dilute systems of instantons, the screening justifies the use of the random gas approximation for finite N_c (unquenched calculations) [4, 6, 7]. In the instanton model of the vacuum the $\eta - \eta'$ mass splitting does not vanish in the large N_c limit.

The screening is unaffected by low temperatures, *i.e.* temperatures for which the pion and kaon mass do not change substantially. As expected, the bulk pressure is meson dominated. The η' contribution to the pressure can be viewed as originating either from a strongly correlated and massive η' at temperature T, or from a free light singlet at temperature T mixed to a four dimensional Coulomb gas with an effective temperature $f(T)$. The latter can be described by a Debye-Huckel equation in four dimensions. At high

temperature substantial changes in the bulk parameters together with strong hadronic correlations make a simple assessment of the instanton contribution to the pressure difficult. The screening, however, is found to decrease at high temperature. Our estimates in matter becomes unreliable at temperatures of the order of $T \sim m_0^4 < N > /V_3$. At these temperatures, a Kosterlitz-Thouless transition in the instanton language may occur, followed by a substantial drop in the topological susceptibility, since $\chi(q = 0) \sim m < \overline{\psi}\psi >\sim 0$. This transition is likely to be $U_A(1)$ restoring and may be related to the chiral restoration transition in QCD. This point can be clarified by studying the static correlations in the topological charge at large spatial separations.

In so far, our attitude has been to constrain the instanton effects by working from the long wavelength limit, where the physics is known. Alternatively, one can try to start from the short wavelength limit, and build up the correlations from the original fermion determinant in the QCD action [4]. We believe that our arguments provide helpful constraints in the long wavelength limit both in the vacuum and in matter.

Acknowledgements

References

[1] D.I. Diakonov and V. Yu. Petrov, Nucl. Phys. **B272** (1986) 457.

[2] E.V. Shuryak, Nucl. Phys. **B302** (1988) 574.

[3] M. A Nowak, J.J.M. Verbaarschot and I. Zahed, Nucl. Phys. **B324** (1989) 1.

[4] E.V. Shuryak and J.J.M. Verbaarschot, SUNY-NTG-92-40; T. Schafer, E.V. Shuryak and J.J.M. Verbaarschot, SUNY-NTG-92-45;

[5] M.C. Chu, J.M. Grandy, S. Huang and J.W. Negele, Phys. Rev. Lett. **70** (1993) 225.

[6] D.I. Diakonov and V.Yu Petrov, LNPI-1153 (1986) unpublished.

[7] M. A Nowak, J.J.M. Verbaarschot and I. Zahed, Phys. Lett. **B217** (1989) 157; R. Alkofer, M. A Nowak, J.J.M. Verbaarschot and I. Zahed, Phys. Lett. **B233** (1989) 205.

[8] D.I. Diakonov and V. Yu. Petrov, Nucl. Phys. **B245** (1984) 259.

[9] J.J.M. Verbaarschot, Nucl. Phys. **B362** (1991) 33; (E) *ibid* **B386** (1992) 236.

[10] G. Greensite, Nucl. Phys. **B249** (1985) 263.

[11] G. 't Hooft, Phys. Rev. Lett., **37** (1976) 8.

[12] S. Samuel, Mod. Phys. Lett. **A7** (1992) 2007; H. Kikuchi and J. Wudka, Phys. Lett. **B284** (1992) 111; N. Dowrick and N. McDougall, Phys. Lett. **B285** (1992) 269.

[13] J.M. Kosterlitz and D.J. Thouless, J. Phys. **C6** (1973) 1181.

[14] C.G. Callan, N. Coote and D.J. Gross, Phys. Rev. **D13** (1976) 1649; M.B. Einhorn , Phys. Rev. **D14** (1976) 3451.

[15] E.V. Shuryak and J.J.M. Verbaarschot, in preparation.

[16] E. Witten, Nucl. Phys. **B156** (1979) 269.

[17] G. Veneziano, Nucl. Phys. **B159** (1979) 461.

[18] I. Sachs and A. Wipf, Helv. Phys. Acta **65** (1992) 652, and references therein.

[19] R. Crewther, Phys. Lett. **B70** (1977) 349; *ibid* **B93** (1980) 75.

[20] A. Smilga, BUTP-92-26.

[21] M.A. Shifman, Phys. Rep. **209** (1991) 341.

[22] M.A. Nowak and I. Zahed, in preparation.

[23] D.I. Diakonov and V.Y. Mirlin, Phys. Lett. **B203** (1988) 299.

[24] M. A Nowak, J.J.M. Verbaarschot and I. Zahed, Nucl. Phys. **B325** (1989) 581.

[25] C. Adami, T. Hatsuda and I. Zahed, Phys. Rev. **D43** (1991) 921.

[26] D. Gross, R. Pisarski and L. Yaffe, Rev. Mod. Phys. **53** (1981) 43.

[27] T.H. Hansson and I. Zahed, Phys. Lett. **B309** (1993) 385.

[28] E.M. Ilgenfritz and E.V. Shuryak, SUNY-NTG-94-4.